

A DIFFERENTIATION THEORY FOR ITÔ'S CALCULUS

HASSAN ALLOUBA

ABSTRACT. A peculiar feature of Itô's calculus is that it is an integral calculus that gives no explicit derivative with a systematic differentiation theory counterpart, as in elementary calculus. So, can we define a pathwise stochastic derivative of semimartingales with respect to Brownian motion that leads to a differentiation theory counterpart to Itô's integral calculus? From Itô's definition of his integral, such a derivative must be based on the quadratic covariation process. We give such a derivative in this note and we show that it leads to a fundamental theorem of stochastic calculus, a generalized stochastic chain rule that includes the case of convex functions acting on continuous semimartingales, and the stochastic mean value and Rolle's theorems. In addition, it interacts with basic algebraic operations on semimartingales similarly to the way the deterministic derivative does on deterministic functions, making it natural for computations. Such a differentiation theory leads to many interesting applications some of which we address in an upcoming article.

1. DIFFERENTIATING SEMIMARTINGALES WITH RESPECT TO BROWNIAN MOTION

One of the greatest twentieth century's discoveries in probability and mathematics is Itô's theory of stochastic integration [5] which, in its simplest form, shows how to integrate certain stochastic processes with respect to Brownian motion (BM). Itô's powerful ideas are still at the heart of some of the most important advances in both pure and applied mathematics sixty years later (stochastic analysis, SDEs, SPDEs, finance, and others). As is well known, Itô's calculus is an integral calculus that gives no explicit derivative and no systematic differentiation theory counterpart, as in elementary calculus. So, the question is: can we define a pathwise stochastic derivative of semimartingales with respect to BM that leads to a differentiation theory counterpart to Itô's integral calculus? Before giving such a derivative, we briefly recall that the essential ingredient in Itô's definition of the integral $\int_0^t X_s dB_s$ of a stochastic process $X = \{X_t; 0 \leq t < \infty\}$ with respect to a BM $B = \{B_t; 0 \leq t < \infty\}$ is Itô's isometry

$$\mathbb{E} \left(\int_0^t X_s dB_s \right)^2 = \mathbb{E} \int_0^t X_s^2 ds,$$

where \mathbb{E} is the usual mathematical expectation. This isometry leads to a definition of the integral with respect to B in terms of one with respect to the quadratic variation of B given by $\langle B, B \rangle_t(\omega) = \langle B \rangle_t(\omega) = t$. Our idea is then to define the derivative dS_t/dB_t of a semimartingale S with respect to a BM B at t as a generalized version of the pathwise stochastic derivative $d\langle S, B \rangle_t(\omega)/d\langle B \rangle_t(\omega) = d\langle S, B \rangle_t(\omega)/dt$ of the covariation of S and B at t with respect to the quadratic variation of B at t . In fact, our derivative covers cases where $d\langle S, B \rangle_t(\omega)/dt$ doesn't exist for all t . We use an integral formulation to define our stochastic derivative (Definition 1.1), which allows for greater applicability: for example if $t \mapsto \langle S, B \rangle_t$ is convex almost surely, then our

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derivative of S with respect to Brownian motion B exists for all $t \in \mathbb{R}_+$ a.s. (see Lemma 2.1). We show that our derivative is an anti Itô integral that has the “desired” properties leading to a fundamental theorem of stochastic calculus (Theorem 2.1), a generalized stochastic chain rule (Theorem 3.1) that includes the case of convex functions acting on continuous semimartingales, and the stochastic mean value and Rolle’s theorems (Lemma 2.1). In addition, it interacts with basic algebraic operations on semimartingales similarly to the way Newton’s deterministic derivative does on deterministic functions (Theorem 3.2), making it natural for computations.

Heuristically, our derivative of a semimartingale S with respect to a BM B is the pathwise “velocity” of S relative to B at each point in time (we call it the B -Brownian velocity of S , see the examples at the end of Section 3). It gives a Brownian path view of the changes in a semimartingale’s path $S(\omega)$ in time by measuring the rate of change of the covariation of S with the Brownian motion B with respect to the quadratic variation of B (time). The sign of our derivative of S with respect to B tells us, path by path, whether S and B are increasing and decreasing together (positive sign) or whether the value of S changes in the opposite direction of changes in the value of B (negative sign). The magnitude of the B -Brownian velocity of S gives us the B -Brownian speed of S .

We believe that this pathwise view, in addition to giving rise to a differentiation theory for Itô’s calculus, is a useful tool in the analysis of SDEs and SPDEs; leading to a new smoothness and regularity theory for solutions of these stochastic equations, which in turns leads to new insights into the behavior of solutions to SDEs and SPDEs relative to their driving noise. It also leads to a rich differential stochastic vector calculus as well as to a new stochastic optimization theory (optimization with respect to the noise). We study these different aspects in more details in [1] and in subsequent articles.

Notation 1.1. *Throughout this article, let $S = \{S_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ be a continuous semimartingale, let $M = \{M_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ and $V = \{V_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ be the continuous local martingale and the continuous process of bounded variation in the decomposition of S , respectively, and let $B = \{B_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ be a standard BM on the same usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ (a usual probability space is one where the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions: right continuity and completeness). Also, let $D\langle S, B \rangle_t \triangleq d\langle S, B \rangle_t/dt$, which we call the strong derivative of S with respect to B . Finally, we denote by λ the Lebesgue measure on \mathbb{R}_+ and by $\mathring{\mathbb{R}}_+$ the set $\mathbb{R}_+ \setminus \{0\}$.*

Definition 1.1 (Derivative of semimartingale with respect to BM). *The stochastic difference and stochastic derivative of S with respect to B at t , respectively, are defined by*

$$(1.1) \quad D_{B_t, h} S_t \triangleq \begin{cases} \frac{3}{2h^3} \int_0^h r [\langle S, B \rangle_{t+r} - \langle S, B \rangle_{t-r}] dr; & 0 < t < \infty, h > 0, \\ \frac{3}{h^3} \int_0^h r \langle S, B \rangle_r dr; & t = 0, h > 0. \end{cases}$$

$$\mathbb{D}_{B_t} S_t \triangleq \lim_{h \rightarrow 0} D_{B_t, h} S_t; \quad 0 \leq t < \infty,$$

whenever this limit exists. We call continuous semimartingales for which the limit in (1.1) exists for all $t \in \mathbb{S} \in \mathcal{B}(\mathbb{R}_+)$ a.s. differentiable with respect to B , on \mathbb{S} , and we denote this class by $\mathcal{S}_B(\mathbb{S})$. The k -th B -derivative of S is defined iteratively in the obvious way, and the class of k -times B -differentiable elements of $\mathcal{S}_B(\mathbb{S})$ is denoted by $\mathcal{S}_B^{(k)}(\mathbb{S})$. If the derivative $d\langle S, B \rangle_t/dt$ exists then

$$(1.2) \quad \mathbb{D}_{B_t} S_t = \frac{d\langle S, B \rangle_t}{dt}$$

(see Theorem 2.1 below), and we call $d\langle S, B \rangle_t / dt$ the strong derivative of S with respect to B at t . We denote the class of continuous semimartingales whose strong B -derivative exists on \mathbb{S} by $\mathcal{S}_B^s(\mathbb{S})$. The class of k -times strongly B -differentiable elements of $\mathcal{S}_B^s(\mathbb{S})$ is denoted by $\mathcal{S}_B^{s,(k)}(\mathbb{S})$.

If the strong B -derivative of S exists at t and if $f \in C^1(\mathbb{R}; \mathbb{R})$ with $f'(x) \neq 0$ for all $x \in \mathbb{R}$ and the map $x \mapsto f'(x)$ is absolutely continuous, we define the strong derivative of S with respect to the semimartingale $S^{(2)} \triangleq f(B)$ at t by

$$(1.3) \quad \mathbb{D}_{S^{(2)}} S_t \triangleq \frac{d\langle S, B \rangle_t}{d\langle S^{(2)} \rangle_t}.$$

We also have the same definition of $\mathbb{D}_{S^{(2)}} S_t$ in the case f is convex and $f'_-(x) \neq 0$ for all $x \in \mathbb{R}$, where $f'_-(x)$ is the left derivative of f at x . The generalized derivative of S with respect to $f(B)$ at t is obtained straightforwardly from (1.1).

Remark 1.1. (a) It follows immediately from our Definition 1.1 of the stochastic derivative process that $\mathbb{D}_B V \equiv 0$ for all processes V of bounded variation on compacts: there changes are “too slow” for the BM to pickup, and they behave like constants in elementary calculus in that their derivative is 0. Additionally, if M and B are independent or orthogonal ($\langle M, B \rangle \equiv 0$); then from the definitions above $\mathbb{D}_B M \equiv 0$: M is independent of or orthogonal to B and therefore it is “unaffected” by changes in B .

(b) We don't use the following observation here in this note; but, we formally think of $D_{B_t, h} S_t$ as the quadratic variation of a generalized stochastic integral which we call the pre-stochastic difference of S with respect to the Brownian motion B :

$$PD_{B_t, h} S_t \triangleq \begin{cases} \sqrt{\frac{3}{2h^3}} \int_0^h \sqrt{r} [\langle S, B \rangle_{t+r} - \langle S, B \rangle_{t-r}]^{\frac{1}{2}} dB_r; & 0 < t < \infty, \ h > 0, \\ \sqrt{\frac{3}{h^3}} \int_0^h \sqrt{r} [\langle S, B \rangle_r]^{\frac{1}{2}} dB_r; & t = 0, \ h > 0. \end{cases}$$

2. FUNDAMENTAL RESULTS

We start with a lemma which expresses our stochastic derivative as a generalized derivative of the covariance process $\langle M, B \rangle$ with respect to time $t = \langle B \rangle_t$ as well as gives us a stochastic mean value theorem (SMVT) and a stochastic Rolle's theorem (SRT).

Lemma 2.1. Let S , M , V , and B be defined as in Notation 1.1 on the same usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

(a) (Stochastic derivative as a generalized derivative of the covariance process) Assume that the one sided derivatives processes $D^+ \langle M, B \rangle$ and $D^- \langle M, B \rangle$ are finite a.s.; i.e.,

$$(2.1) \quad \begin{cases} -\infty < D^\pm \langle M, B \rangle_t = \lim_{\epsilon \rightarrow 0^\pm} \frac{\langle M, B \rangle_{t+\epsilon} - \langle M, B \rangle_t}{\epsilon} < \infty; & 0 < t < \infty \text{ a.s. } \mathbb{P}, \\ -\infty < (D^+ \langle M, B \rangle_t)|_{t=0} < \infty; & t = 0 \text{ a.s. } \mathbb{P} \end{cases}$$

(e.g., if $t \mapsto \langle M, B \rangle_t$ is convex a.s. \mathbb{P}). Then, $S \in \mathcal{S}_B(\mathbb{R}_+)$ and the stochastic derivative of S with respect to B is given by

$$(2.2) \quad \mathbb{D}_{B_t} S_t = \begin{cases} \frac{1}{2} [D^+ \langle M, B \rangle_t + D^- \langle M, B \rangle_t]; & 0 < t < \infty, \text{ a.s. } \mathbb{P}, \\ (D^+ \langle M, B \rangle_t)|_{t=0}; & t = 0, \text{ a.s. } \mathbb{P}. \end{cases}$$

In particular, if (2.2) holds on $\Omega^* \subset \Omega$ (with $\mathbb{P}(\Omega^*) = 1$) and $\langle M, B \rangle(\omega_0)$ is differentiable at t for some $\omega_0 \in \Omega^*$; then $\mathbb{D}_{B_t} S_t(\omega_0) = D\langle M, B \rangle_t(\omega_0)$.

- (b) (SMVT and SRT) Let $\langle M, B \rangle$ be continuous on the closed interval $[a, b] \subset \mathbb{R}_+$ and differentiable on the open interval (a, b) , a.s. \mathbb{P} ; then,

$$(2.3) \quad (\mathbb{D}_{B_t} S_t)|_{t=c}(\omega) = \frac{\langle M, B \rangle_b(\omega) - \langle M, B \rangle_a(\omega)}{b - a},$$

for some random variable $c(\omega) \in (a, b)$ a.s. \mathbb{P} . In particular, if $\langle M, B \rangle_b(\omega) = \langle M, B \rangle_a(\omega)$ a.s. \mathbb{P} ; then $\mathbb{D}_{B_t} S_t|_{t=c}(\omega) = 0$ a.s. \mathbb{P} .

A simple application of the SMVT leads to

Corollary 2.1. Suppose that, for some BM B , $S \in \mathcal{S}_B^s(\mathbb{R}_+)$ with decomposition $S_t = S_0 + V_t + M_t$, $t \geq 0$, a.s. \mathbb{P} .

- (i) If $\mathbb{D}_{B_t} S_t = 0$ for all $t > 0$ a.s. \mathbb{P} . Then, $\langle M, B \rangle \equiv 0$ a.s. \mathbb{P} .
- (ii) If $\mathbb{D}_{B_t} S_t$ does not change sign on (a, b) a.s. \mathbb{P} , then $\langle M, B \rangle$ is monotonic over (a, b) a.s. \mathbb{P} : increasing, nondecreasing, decreasing, or nonincreasing as

$$\mathbb{D}_{B_t} S_t > 0, \mathbb{D}_{B_t} S_t \geq 0, \mathbb{D}_{B_t} S_t < 0, \text{ or } \mathbb{D}_{B_t} S_t \leq 0; \forall t \in (a, b) \text{ a.s. } \mathbb{P},$$

respectively.

- (iii) If

$$|\mathbb{D}_{B_t} S_t| \leq K; \quad a < t < b, \text{ a.s. } \mathbb{P}$$

then $\langle M, B \rangle$ is Lipschitz on (a, b) :

$$|\langle M, B \rangle_t - \langle M, B \rangle_s| \leq K|t - s|; \quad t, s \in (a, b) \text{ a.s. } \mathbb{P}.$$

We state the Fundamental theorem of stochastic calculus for a class of semimartingales that covers Itô SDEs of interest and that can be generalized to cover SPDEs of interest as well:

$$(2.4) \quad S = \left\{ S_t = S_0 + V_t + \int_0^t X_s dB_s, \mathcal{F}_t; \quad 0 \leq t < \infty \right\}$$

where V and B are as in Notation 1.1; and the adapted process $X = \{X_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ is in $L^2([0, t]; \lambda) \forall t > 0$ a.s. \mathbb{P} . We denote by $\mathcal{S}^{\text{SI}_B}$ the class of continuous semimartingales S whose local martingale part M is given by the stochastic integral in (2.4); i.e.,

$$(2.5) \quad M_t = \int_0^t X_s dB_s, \mathcal{F}_t; \quad 0 \leq t < \infty.$$

The elements of $\mathcal{S}^{\text{SI}_B}$ in which the integrand X has a.s. continuous paths form the subclass which we denote by $\mathcal{S}_c^{\text{SI}_B}$. Finally, we denote by $\mathcal{S}_0^{\text{SI}_B}$ and $\mathcal{S}_{c,0}^{\text{SI}_B}$ the subclasses $\mathcal{S}_0^{\text{SI}_B} \subset \mathcal{S}^{\text{SI}_B}$ and $\mathcal{S}_{c,0}^{\text{SI}_B} \subset \mathcal{S}_c^{\text{SI}_B}$ in which $V \equiv 0$ for all of its elements.

We are now ready to present the pathwise fundamental theorem of stochastic calculus (FTSC)

Theorem 2.1 (FTSC). Let $S \in \mathcal{S}_c^{\text{SI}_B}$. Then, $S \in \mathcal{S}_B^s(\mathbb{R}_+)$ —in particular the process $\langle M, B \rangle$ is differentiable for all $t \in \mathbb{R}_+$, a.s. \mathbb{P} —and

- (i) the stochastic derivative process $\mathbb{D}_B S = \{\mathbb{D}_{B_t} S_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ is given by $\mathbb{D}_{B_t} S_t = X_t$ for all $t \in \mathbb{R}_+$, a.s. \mathbb{P} . In particular,

$$(2.6) \quad \mathbb{D}_{B_t} \int_0^t X_s dB_s = X_t; \quad \forall 0 \leq t < \infty, \text{ a.s. } \mathbb{P}.$$

Moreover, if $S, \tilde{S} \in \mathcal{S}_c^{\text{SI}_B}$ with $\mathbb{P} \left\{ M_t = \tilde{M}_t; \forall t \in \mathbb{R}_+ \right\} = 1$; then their stochastic derivative processes are indistinguishable: $\mathbb{P} \left\{ \mathbb{D}_{B_t} S_t = \mathbb{D}_{B_t} \tilde{S}_t; \forall t \in \mathbb{R}_+ \right\} = 1$.

(ii)

$$(2.7) \quad \int_0^t \mathbb{D}_{B_s} S_s dB_s = \tilde{S}_t - \tilde{S}_0 - \tilde{V}_t; \quad \forall 0 \leq t < \infty, \quad \text{a.s. } \mathbb{P}$$

for any $\tilde{S} \in \mathcal{S}_c^{\text{SI}_B}$ whose local martingale part \tilde{M} is indistinguishable from M . In particular; $\int_0^t \mathbb{D}_{B_s} S_s dB_s = S_t - S_0 - V_t$ for all $t \in \mathbb{R}_+$, a.s. \mathbb{P} . Thus, if $S \in \mathcal{S}_{c,0}^{\text{SI}_B}$; then, $\int_0^t \mathbb{D}_{B_s} S_s dB_s = S_t - S_0$ for all $t \in \mathbb{R}_+$, a.s. \mathbb{P} .

Remark 2.1. In contrast to the fundamental theorem of deterministic calculus, part (ii) of Theorem 2.1 involves the additional term of bounded variation on compacts \tilde{V} , unless $S \in \mathcal{S}_{c,0}^{\text{SI}_B}$. Remember however that, a.s. \mathbb{P} , $\mathbb{D}_B \tilde{V} \equiv 0$ by Remark 1.1.

In the case the integrand X is not necessarily continuous we state the following

Theorem 2.2. Assume that $S \in \mathcal{S}^{\text{SI}_B}$; then, a.s. \mathbb{P} , the process $\langle M, B \rangle$ is differentiable for all $t \in \mathbb{R}_+ \setminus Z$ and $\mathbb{D}_B S = \left\{ X_t; t \in \mathring{\mathbb{R}}_+ \setminus Z \right\}$, for some $Z(\omega) \subset \mathbb{R}_+$ with $\lambda(Z) = 0$. If, additionally, the condition (2.1) hold; then we also have

$$(2.8) \quad \mathbb{D}_{B_t} S_t = \begin{cases} \frac{1}{2} \left[D^+ \int_0^t X_s ds + D^- \int_0^t X_s ds \right]; & t \in (0, \infty) \cap Z, \\ \left(D^+ \int_0^t X_s ds \right) \Big|_{t=0}; & t = 0, \end{cases}$$

a.s. \mathbb{P} . If $S, \tilde{S} \in \mathcal{S}^{\text{SI}_B}$ with indistinguishable M and \tilde{M} ; then, a.s. \mathbb{P} , $\mathbb{D}_{B_t} S_t = \mathbb{D}_{B_t} \tilde{S}_t$ for all $t \in \mathring{\mathbb{R}}_+ \setminus O$, for some $O(\omega) \subset \mathbb{R}_+$ with $\lambda(O) = 0$ (we say that $\mathbb{D}_B S$ and $\mathbb{D}_B \tilde{S}$ are almost indistinguishable). In particular, if M is a B -Brownian martingale; then, a.s. \mathbb{P} , $\mathbb{D}_B M = \{Y_t; t \in \mathring{\mathbb{R}}_+ \setminus Z\}$ for some $Z(\omega) \subset \mathbb{R}_+$ with $\lambda(Z) = 0$, where Y is the progressively measurable process such that $\mathbb{E}_{\mathbb{P}} \int_0^t Y_s^2 ds < \infty$ and $\int_0^t Y_s dB_s = M_t$, for all $t \in \mathbb{R}_+$.

3. PATHWISE DERIVATIVE RULES

We now show that our pathwise derivative for Itô's calculus generalizes familiar differentiation rules from deterministic to stochastic calculus, making it useful for computations involving functions of semimartingales and algebraic operations on several semimartingales. We start with our chain rule for stochastic calculus.

Theorem 3.1 (The chain rule of stochastic calculus). (a) Suppose that $f \in C^1(\mathbb{R}; \mathbb{R})$ such that the function $x \mapsto f'(x)$ is absolutely continuous.

(i) Then, a.s. \mathbb{P} , the process $\mathbb{D}_B f(S) = \{\mathbb{D}_{B_t} f(S_t); t \in \mathbb{R}_+\}$ is given by

$$(3.1) \quad \mathbb{D}_{B_t} f(S_t) = \begin{cases} \frac{1}{2} \left[\left(D^+ \int_0^t f'(S_s) d\langle M, B \rangle_s \right) + \left(D^- \int_0^t f'(S_s) d\langle M, B \rangle_s \right) \right]; & 0 < t < \infty, \\ \left(D^+ \int_0^t f'(S_s) d\langle M, B \rangle_s \right) \Big|_{t=0}; & t = 0, \end{cases}$$

whenever the one sided derivatives are finite. If $d\langle M, B \rangle_s = X_{M,B}(s)ds$ and $X_{M,B}$ has continuous paths on \mathbb{R}_+ a.s. \mathbb{P} , then $S \in \mathcal{S}_B^s(\mathbb{R}_+)$ and (3.1) becomes

$$(3.2) \quad \mathbb{D}_{B_t} f(S_t) = f'(S_t) \mathbb{D}_{B_t} S_t = f'(S_t) X_{M,B}(t); \quad 0 \leq t < \infty \quad \text{a.s. } \mathbb{P}.$$

In particular, if $S \in \mathcal{S}_c^{\text{SI}_B}$; then

$$(3.3) \quad \mathbb{D}_{B_t} f(S_t) = f'(S_t) \mathbb{D}_{B_t} S_t; \quad 0 \leq t < \infty \quad \text{a.s. } \mathbb{P}.$$

(ii) Suppose further that $f'(x) \neq 0$ for every $x \in \mathbb{R}$, $d\langle M, B \rangle_s = X_{M,B}(s)ds$ and $X_{M,B}$ has continuous paths on \mathbb{R}_+ a.s. \mathbb{P} , and let $S^{(2)} \triangleq f(B)$. Then, $S, S^{(2)} \in \mathbb{S}_B^s(\mathbb{R}_+)$ and

$$(3.4) \quad \mathbb{D}_{B_t} S_t = \mathbb{D}_{S_t^{(2)}} S_t \cdot \mathbb{D}_{B_t} S^{(2)}(t) \quad 0 \leq t < \infty \text{ a.s. } \mathbb{P}.$$

(b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is only assumed to be convex; then (3.1), (3.2), and (3.3) all hold, replacing $f'(x)$ by the left derivative at x , $f'_-(x)$. Moreover, if in addition the assumptions in part (a)(ii) hold (again replacing $f'(x)$ by $f'_-(x)$), then (3.4) holds.

An interesting question then is when is $\mathbb{D}_B S$ itself a martingale (or a local martingale)? It is for example clear from the above discussion that $\{\mathbb{D}_{B_t}(B_t^2 - t) = 2B_t, \mathcal{F}_t; t \geq 0\}$ is a martingale. The next corollary, which follows as an immediate consequence of Theorem 3.1, gives a sufficient condition.

Corollary 3.1. Suppose $f \in C^1(\mathbb{R}; \mathbb{R})$ such that the function $x \mapsto f'(x)$ is absolutely continuous and $S_t = f(B_t) + V_t$, $t \geq 0$; where B and V are as in Notation 1.1. Then

$$\mathbb{D}_{B_t} S_t = f'(B_t); \quad 0 \leq t < \infty.$$

In particular, $\mathbb{D}_{B_t} S = \{\mathbb{D}_{B_t} S_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale (local martingale) iff the process $\{f'(B_t), \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale (local martingale). If $f : \mathbb{R} \rightarrow \mathbb{R}$ is only assumed to be convex, then $\mathbb{D}_{B_t} S = \{\mathbb{D}_{B_t} S_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale (local martingale) iff $\{f'_-(B_t), \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale (local martingale).

Another immediate consequence of Theorem 3.1 is the power rule for our pathwise derivative

Corollary 3.2 (The power rule of stochastic calculus). Let $S \in \mathbb{S}_B(\mathbb{R}_+)$ with continuous local martingale part M . If $d\langle M, B \rangle_s = X_{M,B}(s)ds$ and $X_{M,B}$ has continuous paths on \mathbb{R}_+ a.s. \mathbb{P} , then $S \in \mathbb{S}_B^s(\mathbb{R}_+)$ and

$$(3.5) \quad \mathbb{D}_{B_t} (S_t)^p = p(S_t)^{p-1} \mathbb{D}_{B_t} S_t; \quad 0 \leq t < \infty, \quad \forall p \geq 1.$$

If in addition $S_t \neq 0$ for every $t \in \mathbb{R}_+$, a.s. \mathbb{P} , then

$$(3.6) \quad \mathbb{D}_{B_t} (S_t)^p = p(S_t)^{p-1} \mathbb{D}_{B_t} S_t; \quad 0 \leq t < \infty, \quad \forall p \in \mathbb{R}.$$

Theorem 3.2 (The sum, product, and ratio rules). Let $S_1, S_2 \in \mathbb{S}_B(\mathbb{R}_+)$ and let $a, b \in \mathbb{R}$ be arbitrary but fixed, then $aS^{(1)} \pm bS^{(2)} \in \mathbb{S}_B(\mathbb{R}_+)$ and

$$(3.7) \quad \mathbb{D}_{B_t} (aS_t^{(1)} \pm bS_t^{(2)}) = a\mathbb{D}_{B_t} S_t^{(1)} \pm b\mathbb{D}_{B_t} S_t^{(2)}; \quad 0 \leq t < \infty, \text{ a.s. } \mathbb{P}.$$

If in addition the continuous local martingale parts $M^{(1)}$ and $M^{(2)}$ of $S^{(1)}$ and $S^{(2)}$, respectively, satisfy

$$(3.8) \quad d\langle M^{(i)}, B \rangle_s = X_{M^{(i)}, B}(s)ds; \quad i = 1, 2,$$

and $X_{M^{(i)}, B}$ has continuous paths on \mathbb{R}_+ a.s. \mathbb{P} for $i = 1, 2$ then $S^{(1)}, S^{(2)}, S^{(1)}S^{(2)} \in \mathbb{S}_B^s(\mathbb{R}_+)$ and

$$(3.9) \quad \mathbb{D}_{B_t} (S_t^{(1)} S_t^{(2)}) = S_t^{(2)} \mathbb{D}_{B_t} S_t^{(1)} + S_t^{(1)} \mathbb{D}_{B_t} S_t^{(2)}; \quad 0 \leq t < \infty, \text{ a.s. } \mathbb{P}.$$

If in addition $S_t^{(2)} \neq 0$ for every $t \in \mathbb{R}_+$, a.s. \mathbb{P} , then $S^{(1)}/S^{(2)} \in \mathbb{S}_B^s(\mathbb{R}_+)$ and

$$(3.10) \quad \mathbb{D}_{B_t} \left(\frac{S_t^{(1)}}{S_t^{(2)}} \right) = \frac{S_t^{(2)} \mathbb{D}_{B_t} S_t^{(1)} - S_t^{(1)} \mathbb{D}_{B_t} S_t^{(2)}}{[S_t^{(2)}]^2}; \quad 0 \leq t < \infty, \text{ a.s. } \mathbb{P}.$$

We now briefly look at the Brownian derivatives (Brownian velocity) of several simple semi-martingales obtained as simple applications of Theorem 2.1, Theorem 3.1, and Theorem 3.2. These examples show that our derivative gives intuitive answers:

(1)

$$\mathbb{D}_{B_t} |B_t| = \begin{cases} 1, & B_t > 0 \\ -1 & B_t \leq 0 \end{cases}$$

I.e., the Brownian speed of $|B_t|$ is 1 for all $t \in \mathbb{R}_+$, and the direction of change of $|B|$ (increase or decrease) relative to B at t is the same as B_t if $B_t > 0$ and is opposite to that of B_t if $B_t < 0$.

(2) Let $S_t = B_t^2 - V_t$, for any process V as in Notation 1.1, then $\mathbb{D}_{B_t} S_t = 2B_t$ for all $t \in \mathbb{R}_+$. So that the Brownian speed of S_t at any $t \in \mathbb{R}_+$ is $2|B_t|$, and the direction of change of S_t (increase or decrease) relative to B at t is the same as B_t if $B_t > 0$ and is opposite to that of B_t if $B_t < 0$.

(3) Let

$$\Xi_t^{X,B} = \exp \left(\int_0^t X_s dB_s - \frac{1}{2} \int_0^t X_s^2 ds \right),$$

and let the adapted process $X = \{X_t, \mathcal{F}_t; t \in \mathbb{R}_+\}$ be continuous (thus in $L^2([0, t]; \lambda) \forall t > 0$ a.s. \mathbb{P}). Then, $\mathbb{D}_{B_t} \Xi_t^{X,B} = X_t \Xi_t^{X,B}$. So that the Brownian speed of the exponential local martingale $\Xi_t^{X,B}$ at any $t \in \mathbb{R}_+$ is $|X_t| \Xi_t^{X,B}$, and the direction of change of $\Xi_t^{X,B}$ (increase or decrease) relative to B at t is the same as B_t if $X_t > 0$ and is opposite to that of B_t if $X_t < 0$.

More applications of our theory presented here (including SDEs and SPDEs) are dealt with in [1] and subsequent articles.

4. PROOFS OF RESULTS

Proof of Lemma 2.1.

(a) Under the assumptions given, the conclusion follows from the definition of $\mathbb{D}_{B_t} S_t$, the facts that $\langle S, B \rangle_t = \langle M, B \rangle_t$ and $\langle M, B \rangle_0 = 0$, and the fact that the generalized derivative

$$(4.1) \quad \mathcal{D}g(t) \triangleq \begin{cases} \lim_{h \rightarrow 0} \frac{3}{2h^3} \int_0^h r [g(t+r) - g(t-r)] dr = \frac{1}{2} [D^+g(t) + D^-g(t)]; & 0 < t < \infty, \\ \lim_{h \rightarrow 0} \frac{3}{h^3} \int_0^h r [g(r) - g(0)] dr = D^+g(0); & t = 0. \end{cases}$$

for any function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ with finite one sided derivatives $D^\pm g(t)$ for $0 < t < \infty$ and with finite right hand derivative at zero $D^+g(0)$. Of course if g is differentiable at t then $\mathcal{D}g(t) = g'(t)$.

(b) Let the assumptions hold on $\Omega^* \subset \Omega$, with $\mathbb{P}(\Omega^*) = 1$, and fix $\omega \in \Omega^*$. Then, by the classical mean value theorem \exists a $c(\omega) \in (a, b) \ni$

$$D\langle M, B \rangle_t|_{t=c(\omega)} = \frac{\langle M, B \rangle_b(\omega) - \langle M, B \rangle_a(\omega)}{b - a}.$$

Since $\langle M, B \rangle$ is assumed differentiable on $(a, b) \forall \omega \in \Omega^*$; then, by part (a), $\mathbb{D}_{B_t} S_t|_{t=c(\omega)} = D\langle M, B \rangle_t|_{t=c(\omega)}$ for all $\omega \in \Omega^*$ and (2.3) is established.

The proof is complete. \square

Proof of Theorem 2.1. (i) Under the assumptions on S , the stochastic difference and stochastic derivative of S with respect to B are a.s. \mathbb{P} given, respectively, by

$$(4.2) \quad \begin{aligned} \mathbb{D}_{B_t, h} S_t &= \frac{3}{2h^3} \int_0^h r \left[\int_0^{t+r} X_s ds - \int_0^{t-r} X_s ds \right] dr; \quad 0 < t < \infty, \\ \mathbb{D}_{B_t, h} S_t &= \frac{3}{2h^3} \int_0^h r \left[\int_0^r X_s ds \right] dr; \quad t = 0, \\ \mathbb{D}_{B_t} S_t &= \lim_{h \rightarrow 0} \mathbb{D}_{B_t, h} S_t = X_t; \quad 0 \leq t < \infty. \end{aligned}$$

where the last equality follows by letting $g(t) = \int_0^t X_s ds$ and using the fundamental theorem of classical calculus along with continuity of X and the fact (4.1). Now, if $S, \tilde{S} \in \mathcal{S}_c^{\text{SI}_B}$ with $M \equiv \tilde{M}$ a.s. \mathbb{P} , then $\int_0^t (X_s - \tilde{X}_s)^2 ds = 0 \forall t \in \mathbb{R}_+$ a.s. \mathbb{P} ; which, by the continuity of X and \tilde{X} and the first part of the proof, implies that $\mathbb{D}_B S \equiv X \equiv \tilde{X} \equiv \mathbb{D}_B \tilde{S}$ a.s. \mathbb{P} .

(ii) By part (i) and (2.4) and (2.5) we have $\mathbb{D}_B S \equiv X$ a.s. \mathbb{P} , from which (2.7) follows. The rest of the assertions follow immediately, completing the proof. \square

Proof of Theorem 2.2. The proof proceeds exactly as in the proof of part (i) of Theorem 2.1, taking into account the noncontinuity of X and using the fundamental theorem of Lebesgue calculus (e.g., see Theorem 10 on p. 107 in [6]) and (2.2) in Lemma 2.1. \square

Proof of Theorem 3.1.

(a) If the function $x \mapsto f'(x)$ is absolutely continuous, then f'' exists Lebesgue-almost everywhere and we have that the Itô formula for $f(S_t)$ is given by

$$(4.3) \quad f(S_t) = f(S_0) + \int_0^t f'(S_s) [dM_s + dV_s] + \frac{1}{2} \int_0^t f''(S_s) d\langle M \rangle_s; \quad t \in \mathbb{R}_+, \text{ a.s. } \mathbb{P}.$$

(i) Using Itô's rule (4.3), Remark 1.1 (a) ($\mathbb{D}_{B_t} U \equiv 0$ for any continuous process of bounded variations on compacts) along with the linearity of the cross variation process, and Theorem 2.1 we have a.s. \mathbb{P}

$$(4.4) \quad \begin{aligned} \mathbb{D}_{B_t} f(S_t) &= \mathbb{D}_{B_t} \left\{ f(S_0) + \int_0^t f'(S_s) [dM_s + dV_s] + \frac{1}{2} \int_0^t f''(S_s) d\langle M \rangle_s \right\} \\ &= \begin{cases} \frac{1}{2} \left[D^+ \int_0^t f'(S_s) d\langle M, B \rangle_s + D^- \int_0^t f'(S_s) d\langle M, B \rangle_s \right]; & 0 < t < \infty, \\ \left(D^+ \int_0^t f'(S_s) d\langle M, B \rangle_s \right) \Big|_{t=0}; & t = 0 \end{cases} \end{aligned}$$

whenever the one sided derivatives are finite. If $d\langle M, B \rangle_s = X_{M,B}(s)ds$ and $X_{M,B}$ has continuous paths a.s. \mathbb{P} ; then it follows from (4.4) and the fundamental theorem of classical calculus that $\mathbb{D}_{B_t} f(S_t) = f'(S_t)X_{M,B}(t)$. $t \in \mathbb{R}_+$ a.s. \mathbb{P} . Also, a.s. \mathbb{P}

$$X_{M,B}(t) = \frac{d}{dt} \int_0^t X_{M,B}(s)ds = \frac{d}{dt} \int_0^t d\langle M, B \rangle_s = \mathbb{D}_{B_t} S_t; t \in \mathbb{R}_+,$$

so that (3.2) follows. If $S \in \mathcal{S}_c^{\text{SI}_B}$, then by the same argument above we get that $\mathbb{D}_{B_t} f(S_t) = f'(S_t)X_t = f'(S_t)\mathbb{D}_{B_t} S_t$ for all $t \in \mathbb{R}_+$ a.s. \mathbb{P} and (3.3) follows.

(ii) From part (i) we have $\mathbb{D}_{B_t} S^{(2)}(t) = \mathbb{D}_{B_t} f(B_t) = f'(B_t)$. Now, using Itô's formula

$$(4.5) \quad \begin{aligned} \frac{d\langle S^{(2)} \rangle_t}{dt} &= \frac{d}{dt} \left\langle f(B_0) + \int_0^\cdot f'(B_s) dB_s + \frac{1}{2} \int_0^\cdot f''(B_s) ds \right\rangle_t \\ &= [f'(B_t)]^2 \end{aligned}$$

Also, we have

$$(4.6) \quad \begin{aligned} \frac{d\langle S, S^{(2)} \rangle_t}{dt} &= \frac{d}{dt} \left[\int_0^t f'(B_s) d\langle M, B \rangle_s \right] \\ &= \frac{d}{dt} \left[\int_0^t f'(B_s) X_{M,B}(s) ds \right] = f'(B_t) X_{M,B}(t) \end{aligned}$$

So that

$$(4.7) \quad \mathbb{D}_{S_t^{(2)}} S_t = \frac{d\langle S, S^{(2)} \rangle_t}{d\langle S^{(2)} \rangle_t} = \frac{f'(B_t) X_{M,B}(t)}{[f'(B_t)]^2}$$

from which it follows that

$$\mathbb{D}_{S_t^{(2)}} S_t \cdot \mathbb{D}_{B_t} S^{(2)}(t) = \frac{f'(B_t) X_{M,B}(t)}{[f'(B_t)]^2} \cdot f'(B_t) = X_{M,B}(t) = \frac{d\langle M, B \rangle_t}{dt} = \mathbb{D}_{B_t} S_t$$

as claimed.

(b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is only assumed to be convex, then from standard results in Itô's calculus we get that

$$(4.8) \quad f(S_t) = f(S_0) + \int_0^t f'_-(S_s) [dM_s + dV_s] + \frac{1}{2} \Gamma_t^f; \quad t \in \mathbb{R}_+, \text{ a.s. } \mathbb{P},$$

where Γ^f is a continuous increasing process and $f'_-(x)$ is the left derivative of f at x . The desired results in this convex case then follow by following the same steps in the arguments above, replacing the Itô formula (4.3) by (4.8) and replacing f' by f'_- throughout.

The proof is complete. \square

Proof of Theorem 3.2. We only need to prove the multiplication and ratio rules (3.9) and (3.10), respectively, as the addition/subtraction rule is clear by the linearity of the cross variation process. We start by proving the multiplication rule (3.9). To this end, let $f(x, y) = xy$ for $x, y \in \mathbb{R}$. Applying Itô's formula for functions of several continuous semimartingales (since all the

$D_i f(S_s^{(1)}, S_s^{(2)})$ and $D_{ij} f(S_s^{(1)}, S_s^{(2)})$ are continuous) we get:

$$\begin{aligned}
 \mathbb{D}_{B_t} \left(S_t^{(1)} S_t^{(2)} \right) &= \mathbb{D}_{B_t} f(S_t^{(1)}, S_t^{(2)}) = \mathbb{D}_{B_t} f(S_0^{(1)}, S_0^{(2)}) \\
 &+ \mathbb{D}_{B_t} \left[\sum_{i=1}^2 \int_0^t D_i f(S_s^{(1)}, S_s^{(2)}) dS_s^{(i)} + \frac{1}{2} \sum_{0 \leq i, j \leq 1} \int_0^t D_{ij} f(S_s^{(1)}, S_s^{(2)}) d\langle S^{(i)}, S^{(j)} \rangle_s \right] \\
 &= \mathbb{D}_{B_t} \left[\int_0^t S_s^{(2)} dS_s^{(1)} + \int_0^t S_s^{(1)} dS_s^{(2)} \right] \\
 (4.9) \quad &= \mathbb{D}_{B_t} \left[\int_0^t S_s^{(2)} dM_s^{(1)} + \int_0^t S_s^{(1)} dM_s^{(2)} \right] \\
 &= \frac{d}{dt} \left[\int_0^t S_s^{(2)} X_{M^{(1)}, B}(s) ds + \int_0^t S_s^{(1)} X_{M^{(2)}, B}(s) ds \right] \\
 &= S_t^{(2)} X_{M^{(1)}, B}(t) + S_t^{(1)} X_{M^{(2)}, B}(t) \\
 &= S_t^{(2)} \mathbb{D}_{B_t} S_t^{(1)} + S_t^{(1)} \mathbb{D}_{B_t} S_t^{(2)}; \quad 0 \leq t < \infty, \text{ a.s. } \mathbb{P},
 \end{aligned}$$

where we have used Remark 1.1 (a); the assumption on $d\langle M^{(i)}, B \rangle_s$ for $i = 1, 2$; the fundamental theorem of deterministic calculus; and the fact that $\mathbb{D}_{B_t} S_t^{(i)} = X_{M^{(i)}, B}(t)$ for $i = 1, 2$.

Now, for the ratio rule (3.10) we can either use the product rule in conjunction with the power rule to show that under the additional assumption $(S_t^{(2)} \neq 0 \text{ for every } t \in \mathbb{R}_+, \text{ a.s. } \mathbb{P})$, we have

$$\begin{aligned}
 \mathbb{D}_{B_t} \left(S_t^{(1)} \left[S_t^{(2)} \right]^{-1} \right) &= \left[S_t^{(2)} \right]^{-1} \mathbb{D}_{B_t} S_t^{(1)} - S_t^{(1)} \left[S_t^{(2)} \right]^{-2} \mathbb{D}_{B_t} S_t^{(2)} \\
 &= \frac{S_t^{(2)} \mathbb{D}_{B_t} S_t^{(1)} - S_t^{(1)} \mathbb{D}_{B_t} S_t^{(2)}}{\left[S_t^{(2)} \right]^2}; \quad 0 \leq t < \infty, \text{ a.s. } \mathbb{P}.
 \end{aligned}$$

Alternatively, we can let $g(x, y) = x/y$ for every $x, y \in \mathbb{R}$ such that $y \neq 0$ and proceed exactly as in (4.9), replacing $f(x, y)$ by $g(x, y)$ to get (3.10). \square

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Department of Mathematical Sciences, Kent State University, Kent, OH 44242
 Phone: (330) 672-9028 email: allouba@math.kent.edu